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6D supersymmetry, projective superspace & 4D, $\mathcal{N}=1$ superfields*

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Abstract: In this note, we establish the formulation of 6D, $\mathcal{N}=1$ hypermultiplets in terms of 4D chiral-nonminimal (CNM) scalar multiplets. The coupling of these to 6D, $\mathcal{N}=1$ Yang-Mills multiplets is described. A 6D, $\mathcal{N}=1$ projective superspace formulation is given in which the above multiplets naturally emerge. The covariant superspace quantization of these multiplets is studied in details.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Superspaces, Field Theories in Higher Dimensions.

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Contents

1. Introduction

Six dimensions is the highest one in which supersymmetric multiplets possessing states of maximum helicity one-half exist. The 6D, $\mathcal{N} = 1$ hypermultiplet can be subjected to dimensional reduction to obtain a 5D, $\mathcal{N} = 1$ hypermultiplet, a 4D, $\mathcal{N} = 2$ hypermultiplet, a 3D, $\mathcal{N} = 4$ hypermultiplet, a 2D, $\mathcal{N} = 4$ hypermultiplet and a 1D, $\mathcal{N} = 8$ hypermultiplet. It is known in these lower dimensions additional hypermultiplets unrelated to this chain of reductions also appear. These other hypermultiplets are by definition "twisted" versions of the 'standard' hypermultiplet that descends from six dimensions. In fact, it has been conjectured [1] the number $N_{\mathrm{H.\,M.}}(D)$ of *distinct* on-shell hypermultiplet representations in a spacetime with bosonic dimension D obeys the rule

$$N_{\mathrm{H.\,M.}}(D) \;=\; 2^{5\,-\,D} \tag{1.1}$$

for $D \leq 5$ and by definition $N_{\mathrm{H.\,M.}}(6) = 1$. The origin of this formula is at present not understood. However, there has in more recent times begun to emerge evidence this is related to the representation theory of certain Clifford algebras and K-theory [2].

So the study of the 6D, $\mathcal{N} = 1$ hypermultiplet has diverse applications in many contexts. In the following, we shall show there are two possible 4D, $\mathcal{N} = 1$ formulations for the 6D hypermultiplets. One of these involves using a pair of chiral superfields (the CC formulation, for example was used in the work of [3]) while the other involves one chiral superfield and one complex linear superfield (the CNM formulation). These two formulations are related by a duality transformation.

As long as one is concentrated only upon classical considerations (with the possible exception on issues of the vacuum state) both formulations are equivalent. However, if applications involve quantum mechanical considerations, there are certain advantages of

JHEP05(2006)051

the CNM formulation. In the CC formulation it is possible to use 4D, $\mathcal{N} = 1$ supergraph techniques. In the CNM formulation this can be augmented by the use of projective supergraph techniques'[4 – 6]. The point is projective supergraphs are more efficient calculational techniques that manifest more possible cancellations between different fields than can be seen with the use of the supergraphs associated with ordinary superspace. Concerning $\mathcal{N} = 2$ covariance, an alternative approach would be use of the powerful harmonic superspace [7]. Classically, the two approaches are closely linked, but there is concrete evidence that only the quantum calculations in projective superspace reduce naturally to those of $\mathcal{N} = 1$ superspace. Thus in any intricate quantum calculation, the CNM approach seems most likely the superior one to employ which can be directly compared to $\mathcal{N} = 1$ computations. We also have focused our attention only on projective superspace instead of harmonic superspace as this insure the absence of harmonic singularities [8] known to occur at one and two loops. Although these have been resolved, it seems likely order-by-order this must be implemented to have any ambiguity [9] removed.

With this in mind, a primary purpose of this work is to carry out the quantization of the CNM formulation of the 6D $\mathcal{N} = 1$ hypermultiplet. As is well known, the CNM pair possesses a natural extension to projective superspace. Thus as part of our presentation we include a discussion of the implications of the projective superspace for the present considerations. The paper is organized as follows.

Using the old suggestion by Siegel [10], the first section describes the use of ordinary 4D $\mathcal{N} = 1$ superspace to describe free 6D $\mathcal{N} = 1$ hypermultiplets. Both CC and CNM descriptions are given for the free hypermultiplet. The 4D formalism is shown capable of describing both 6D (1,0) and (0,1) Weyl spinors which can occur within hypermultiplets. Duality transformations are shown to exist between the 6D theories, in complete analogy to the lower dimensional cases where a $\mathcal{N} = 1$ ordinary superspace formalism can be used.

The next section describes the coupling to the non-Abelian Yang-Mills supermultiplet. This is carried out for both CC and CNM systems. The superspace geometry of the vector multiplets is given and the connections for all directions of the six dimensional manifold are shown. It is again shown the 4D formalism is capable of describing both 6D (1,0) and (0,1) Weyl spinors which can occur within vector multiplets. A restriction is observed to require a vector multiplet containing one 6D Weyl spinor can *only* be coupled to a hypermultiplet containing the opposite 6D Weyl spinor.

The following section discusses the quantization of the 6D hypermultiplet and vector multiplet. It is shown these can be constructed by previous known procedures which have been applied to four dimensions. In the CNM case, an infinite tower of ghosts are found to decouple.

The subsequent section addresses the problem of embedding the 6D CNM description of the hypermultiplet and vector multiplet in a projective superspace. This is achieved and it is shown the previously known structures from the case of 4D remain intact for 6D. Tropical and polar multiplets are found. The embedding of the coupled hypermultiplet/Yang-Mills multiplet system is presented. The quantization of the polar multiplet is discussed and its propagator inferred by exploiting the analogies with the 4D case.

We close with a set of concluding remarks, perspectives and include an appendix explaining our notations and conventions with regard to six dimensions.

2. 6D, $\mathcal{N} = 1$ hypermultiplets

In this section we discuss two alternative partially on-shell descriptions of the six-dimensional $\mathcal{N} = 1$ hypermultiplet [11] obtained by using chiral-chiral (CC) and chiral-nonminimal (CNM) multiplets, whose definitions are inspired by their four-dimensional analogues. Following closely the approach to 10D $\mathcal{N} = 1$ SYM of [12], the CC formulation has been already introduced [3]. A formulation in terms of chiral-nonminimal multiplets (CNM) is proposed in this note as a new contribution to the literature.

We remind our readers in six dimensions the physical degrees of freedom of a $\mathcal{N} = 1$ hypermultiplet are described by two complex scalars and a 6D Weyl spinor [11]. Since in six dimensions there are two different kinds of Weyl spinors, we define $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (0, 1)$ hypermultiplets, depending on the 6D chirality of the Weyl spinor (we refer to the appendix for the spinor notations we adopt). An on-shell description in terms of 4D multiplets must correctly describe the six dimensional dynamics of these degrees of freedom.

Following the approach used previously [3, 12] to describe 10D $\mathcal{N} = 1$ SYM and theories in D-dimensions with $5 \leq \mathrm{D} \leq 10$, we use a formalism explicitly covariant under 4D supersymmetry. We parametrize the six-dimensional spacetime by bosonic coordinates x_i, $i = 0, \cdots, 5$. The first four x_i, together with the grassmannian coordinates $(\theta_\alpha, \overline{\theta}_{\dot\alpha})$ describe the ordinary four-dimensional $\mathcal{N} = 1$ superspace (we use 4D notations and conventions of [13]). Defining the complex coordinates

$$\begin{aligned} z &\equiv \frac{1}{2}(x_4 + ix_5)\,, \qquad \partial \equiv \frac{\partial}{\partial z} = \partial_4 - i\partial_5\,; \\ \overline{z} &\equiv \frac{1}{2}(x_4 - ix_5)\,, \qquad \overline{\partial} \equiv \frac{\partial}{\partial \overline{z}} = \partial_4 + i\partial_5\,, \end{aligned} \tag{2.1}$$

the algebra of supercovariant derivatives is the ordinary one $\{D_\alpha, \bar{D}_{\dot\alpha}\} = i(\sigma^a)_{\alpha\dot\alpha}\partial_a$, $a = 0, 1, 2, 3$ supplemented by the extra conditions $[D_\alpha, \partial] = [\bar{D}_{\dot\alpha}, \partial] = [D_\alpha, \overline{\partial}] = [\bar{D}_{\dot\alpha}, \overline{\partial}] = [\partial, \overline{\partial}] = 0$.

The on-shell description of the 6D hypermultiplet in terms of chiral multiplets [3] is accomplished by the introduction of two chiral superfields $\Omega_\pm(x_i, \theta_\alpha, \bar\theta_{\dot\alpha})$

$$\overline{D}_{\dot\alpha}\Omega_\pm = 0\,, \qquad D_\alpha\overline{\Omega}_\pm = 0\,, \tag{2.2}$$

which realize linearly the 4D, $\mathcal{N} = 1$ supersymmetry and whose components are functions of the six bosonic coordinates. The action

$$\begin{aligned} S_{CC} = &\int d^6x\, d^4\theta \Big[\ \overline{\Omega}_+\,\Omega_+ \ + \ \overline{\Omega}_-\,\Omega_- \ \Big] \ + \ \int d^6x\, d^2\theta \Big[\ \Omega_+\,\partial\,\Omega_- \ \Big] \\ &+ \ \int d^6x\, d^2\overline{\theta} \Big[\ \overline{\Omega}_+\,\overline{\partial}\,\overline{\Omega}_- \ \Big]\,, \end{aligned} \tag{2.3}$$

when reduced to components and with auxiliary fields integrated out by the use of their algebraic equations of motion describes correctly the free propagation of the physical degrees of freedom of the $(1,0)$ hypermultiplet. In fact, defining component fields via

$$A_\pm = \Omega_\pm| \,, \qquad \psi^\alpha_\pm = D^\alpha \Omega_\pm| \,, \qquad F_\pm = D^2 \Omega_\pm| \,, \tag{2.4}$$

and eliminating the auxiliary fields from the action (2.3), we obtain

$$S^0_{CC} = \int d^6x \Big[\, \overline{A}_+ \Box_6 A_+ + \overline{A}_- \Box_6 A_- - \overline{\psi}^{\dot\alpha}_+ i\partial_{\alpha\dot\alpha} \psi^\alpha_+ - \overline{\psi}^{\dot\alpha}_- i\partial_{\alpha\dot\alpha} \psi^\alpha_- - \psi^\alpha_- \partial \, \psi_{+\alpha} - \overline{\psi}^{\dot\alpha}_- \overline{\partial} \, \overline{\psi}_{+\dot\alpha} \, \Big] \,, \tag{2.5}$$

where $\Box_6 \equiv \partial^\mu \partial_\mu = \Box_4 + \partial\overline{\partial}$ is the D'Alambertian operator in six dimensions.

This action describes the free dynamics of two complex scalars $A_\pm$ and a 6D Weyl spinor $(\psi^\alpha_+ \,, \ \overline{\psi}^{\dot\alpha}_-)$, as can easily be inferred by comparing the fermionic part of (2.5) with the action (A.7) in the appendix for a free 6D $(1,0)$ Weyl spinor.

An alternative description can be given in terms of a CNM multiplet. To this end, we introduce a pair of superfields (Φ, Σ) whose covariant definitions are inspired by the four dimensional chiral and complex linear superfields [13 – 15]

$$\overline{D}_{\dot\alpha} \Phi = 0 \,, \qquad D_\alpha \overline{\Phi} = 0 \,, \qquad \overline{D}^2 \Sigma = \partial \, \Phi \,, \qquad D^2 \overline{\Sigma} = \overline{\partial} \, \overline{\Phi} \,, \tag{2.6}$$

In analogy with the four dimensional case we define the components fields as

$$\begin{array}{lll} A = \Phi| \,, & \psi_\alpha = D_\alpha \Phi| \,, & F = D^2 \Phi| \,, \\ B = \Sigma| \,, & \overline{\zeta}_{\dot\alpha} = \overline{D}_{\dot\alpha} \Sigma| \,, & H = D^2 \Sigma| \,, \\ \rho_\alpha = D_\alpha \Sigma| \,, & p_{\alpha\dot\alpha} = \overline{D}_{\dot\alpha} D_\alpha \Sigma| \,, & \overline{\beta}_{\dot\alpha} = \dfrac{1}{2} D^\alpha \overline{D}_{\dot\alpha} D_\alpha \Sigma| \,. \end{array} \tag{2.7}$$

These are functions of the 6D spacetime coordinates.

The action describing the free propagation of these superfields is

$$S_{CNM} = \int d^6x d^4\theta \Big[\, \overline{\Phi}\Phi - \overline{\Sigma}\Sigma \, \Big] \,. \tag{2.8}$$

Due to the constraints (2.6), when reduced to components it takes the form

$$\begin{aligned} S_{CNM} \ = \int d^6x \Big[\ & \overline{A} \Box_4 A \ + \ \overline{B} \Box_4 B \ - \ i \, \overline{\psi}^{\dot\alpha} \partial_{\alpha\dot\alpha} \psi^\alpha \ - \ i \, \overline{\zeta}^{\dot\alpha} \partial_{\alpha\dot\alpha} \zeta^\alpha \\ & - \ \zeta^\alpha \partial \, \psi_\alpha \ - \ \overline{\zeta}^{\dot\alpha} \overline{\partial} \, \overline{\psi}_{\dot\alpha} \ + \ \overline{A} \, \overline{\partial} \partial \, A \ - \ B \, \overline{\partial} \, \overline{F} \ - \ \overline{B} \, \partial \, F \\ & + \ \overline{F} F \ - \ \overline{H} H \ + \ \beta^\alpha \rho_\alpha \ + \ \overline{\rho}^{\dot\alpha} \overline{\beta}_{\dot\alpha} \ - \ \overline{p}^{\alpha\dot\alpha} p_{\alpha\dot\alpha} \ \Big] \,. \end{aligned} \tag{2.9}$$

The auxiliary fields $F, H, \beta_\alpha, \rho_\alpha, p_{\alpha\dot\alpha}$ and their hermitian conjugates satisfy algebraic equations of motion and can be eliminated. The result is

$$S^0_{CNM} = \int d^6x \Big[\, \overline{B} \Box_6 B + \overline{A} \Box_6 A - \overline{\psi}^{\dot\alpha} i\partial_{\alpha\dot\alpha} \psi^\alpha - \overline{\zeta}^{\dot\alpha} i\partial_{\alpha\dot\alpha} \zeta^\alpha - \zeta^\alpha \partial \, \psi_\alpha - \overline{\zeta}^{\dot\alpha} \overline{\partial} \, \overline{\psi}_{\dot\alpha} \, \Big] \,. \tag{2.10}$$

Here again we see the free dynamics of a 6D $\mathcal{N} = (1,0)$ massless hypermultiplet which has as physical degrees of freedom the two complex scalars A, B and the $(1,0)$ 6D Weyl spinor $(\psi^\alpha \,, \ \overline{\zeta}^{\dot\alpha})$. Our results in (2.6-2.10) can also be seen to be a dimensional oxidation of the recent CNM description [16] of the 5D, $\mathcal{N} = 1$ hypermultiplet.

There are some interesting and subtle differences in the two formulations. To reach (2.5) from (2.3) both auxiliary fields F_+ and F_- were removed via their equations of motion. The result of this is to insure the six dimensional D'Alambertian operator appears for $A_\pm$ in (2.3). Something rather different occurs in deriving (2.10) from (2.9). The six dimensional D'Alambertian is already present for A even prior to the elimination of any auxiliary field.

A well-known fact in four dimensions is chiral and complex-linear massless superfields are dual to each other [13]. The CC and CNM 6D massless hypermultiplets introduced above are the analogues of the 4D chiral and complex-linear massless superfields, respectively.[1] We then expect the 4D duality of the two multiplets to be extended to the 6D hypermultiplets.

This is easily implemented by introducing the following action

$$\int d^6x d^4\theta \Big[\overline{\Psi}\Psi - \overline{\eta}\eta + Y(\overline{D}^2\eta - \partial\Psi) + \overline{Y}(D^2\overline{\eta} - \overline{\partial}\,\overline{\Psi}) \Big] \,, \tag{2.11}$$

where Ψ ($\overline{\Psi}$) is (anti)chiral and η, $\overline{\eta}$, Y, $\overline{Y}$ are unconstrained complex superfields. The superfields Y and $\overline{Y}$ act as complex Lagrange multipliers for the nonminimal part of the CNM constraints (2.6). In fact, integrating out Y and $\overline{Y}$, the action (2.11) reduces to (2.8) with $\Phi \equiv \Psi$ and $\Sigma \equiv \eta$ which are now constrained to satisfy the conditions (2.6). On the other hand, if we integrate out η and $\overline{\eta}$ using the equations of motion $\eta = D^2\overline{Y}$, $\overline{\eta} = \overline{D}^2 Y$, and define the chiral superfields $\Omega_- \equiv \overline{D}^2 Y$, $\overline{\Omega}_- \equiv D^2\overline{Y}$, $\Omega_+ \equiv \Psi$, $\overline{\Omega}_+ \equiv \overline{\Psi}$, the action (2.11) reduces to (2.3) and describes a CC hypermultiplet.

We note this kind of duality is *solely* due to the 4D superspace structure which we use to define the multiplets and it should not be affected by the spacetime dimension in which we are working. As important it should be noted this duality would likely *not* exist in a manifestly 6D $\mathcal{N} = 1$ formulation. So one or the other of these two distinct but duality-related formulations may be preferred from this perspective.

Therefore, we expect in six dimensions more general duality patterns. For example, in [14] the most general class of 4D CNM models with constraints $\overline{D}^2\Sigma^a = Q^a(\Phi^b)$ was proposed, where Σ_a are nonminimal superfields and Q^a are holomorphic functions of a set of chirals Φ^b. Using a simple generalization of the ordinary duality transformations, it is possible to prove the four dimensional CNM models described by the action

$$\int d^4x d^4\theta \Big[\overline{\Phi}_b \Phi^b - \overline{\Sigma}_a \Sigma^a \Big] \tag{2.12}$$

are dual to CC models with action

$$\int d^4x d^4\theta \Big[\overline{\Omega}_{+b}\Omega_+^b + \overline{\Omega}_-^a \Omega_{-a} \Big] + \left\{ \int d^4x d^2\theta\, \Omega_{-a} Q^a(\Omega_+^b) + \text{h.c.} \right\}. \tag{2.13}$$

It may be possible to extend this general duality to six dimensional models. As the dimension of spacetime is varied, the class of functions Q^a which are used can also be subjected to different constraints. For example, in four dimensions the Q-functions may be used to

[1] Note the constraint for the nonminimal multiplet Σ as given in (2.6) is modified respect to the ordinary $\bar{D}^2\Sigma = 0$, in analogy to the 4D CNM generalizations proposed in [14] and further studied in [18]. However, as discussed in [5], the duality properties between a pair of massive chirals and a pair of a chiral and a nonminimal superfield survive the more general case $\overline{D}^2\Sigma = m\Phi$.

introduce quartic component-level self-couplings among the scalars. It is unlikely that such interactions are allowed in dimensions greater than four. Still the Q-functions are useful in other ways. For the purposes of this work the relevant choice is $Q = \partial\Phi$ which gives the 6D, $\mathcal{N} = (1,0)$ hypermultiplet.

To conclude this section we note the descriptions of the $\mathcal{N} = (1,0)$ hypermultiplet given above can be easily implemented in the case of a hypermultiplet with opposite chirality. To describe $\mathcal{N} = (0,1)$ hypermultiplets it is only necessary to exchange ∂ with $-\overline{\partial}$ in all the previous formulae. In fact, in this way the scalar parts of the actions (2.5), (2.10) do not change, whereas the spinors built from Ω_-, Ω_+ and Φ, Σ describe the free dynamics of a 6D $(0,1)$ Weyl spinor as in (A.8) (once again in the appendix), thus giving the correct physical content of a $\mathcal{N} = (0,1)$ hypermultiplet.

3. Coupling hypermultiplets to SYM

On-shell formulations of higher dimensional supersymmetric Yang-Mills theories, based on a formalism which keeps 4D supersymmetry manifest, have been considered in literature for the 10D case [12] and successively for any dimension D, $5 \leq D \leq 10$ [3]. We will review the results for the six dimensional case since we are eventually interested to minimally couple matter described by the 6D, CNM hypermultiplets introduced in the previous section.

Superspace covariant derivatives and field strengths can be constructed [12] in terms of a real prepotential V and a (anti)chiral superfield Ω ($\overline{\Omega}$). These are functions of the six bosonic coordinates and have an ordinary expansion in $(\theta_\alpha, \bar{\theta}_{\dot{\alpha}})$, so realizing a representation of 4D supersymmetry. They belong to the adjoint representation of the gauge group, $V = V^i T_i$, $\Omega = \Omega^i T_i$ where T_i are the group generators. They are subjected to the gauge transformations

$$\begin{aligned} e^{V'} &= e^{i\overline{\Lambda}} e^V e^{-i\Lambda}\,, \\ \Omega' &= e^{i\Lambda}\Omega e^{-i\Lambda} + i e^{i\Lambda}(\,\partial e^{-i\Lambda}\,)\,, \qquad \overline{\Omega}' = e^{i\overline{\Lambda}}\overline{\Omega} e^{-i\overline{\Lambda}} + i e^{i\overline{\Lambda}}(\,\overline{\partial} e^{-i\overline{\Lambda}}\,)\,, \end{aligned} \tag{3.1}$$

where Λ is a 4D chiral superfield depending also on the two extra coordinates $z, \bar{z}$.

In the chiral representation covariant derivatives are given by

$$\begin{aligned} &\overline{\nabla}_{\dot{\alpha}} = \overline{D}_{\dot{\alpha}} \qquad , \qquad \nabla_\alpha = D_\alpha - i\Gamma_\alpha = e^{-V} D_\alpha e^V\,, \qquad \nabla_{\alpha\dot{\alpha}} = -i\{\nabla_\alpha, \overline{\nabla}_{\dot{\alpha}}\}\,, \\ &\nabla_z = \partial - i\Gamma_z = \partial - i\Omega\,, \qquad \overline{\nabla}_{\overline{z}} = \overline{\partial} - i\overline{\Gamma}_{\overline{z}} = e^{-V}(\,\overline{\partial} - i\overline{\Omega}\,)e^V\,, \end{aligned} \tag{3.2}$$

and transform covariantly under the gauge transformations (3.1) ($\nabla_A \to e^{i\Lambda}\nabla_A e^{-i\Lambda}$, where $A = (\alpha, \dot{\alpha}, \alpha\dot{\alpha}, z, \bar{z})$). This geometric set-up is the generalization to six dimensions of the chiral representation of the ordinary $\mathcal{N} = 1$ superspace gauge covariant derivatives in four dimensions. The superfields Ω and $\overline{\Omega}$ play the role of connections associated with the two extra derivatives ∂, $\overline{\partial}$.

The covariant derivatives satisfy the constraints

$$\begin{aligned} F_{\alpha\beta} &= i\{\nabla_\alpha, \nabla_\beta\} = 0\,, \qquad & F_{\dot{\alpha}\dot{\beta}} &= i\{\overline{\nabla}_{\dot{\alpha}}, \overline{\nabla}_{\dot{\beta}}\} = 0\,, \\ F_{\alpha\overline{z}} &= i[\nabla_\alpha, \overline{\nabla}_{\overline{z}}] = 0\,, & F_{\dot{\alpha}z} &= i[\overline{\nabla}_{\dot{\alpha}}, \nabla_z] = 0\,, \end{aligned} \tag{3.3}$$

and give the set of non-trivial field strengths

$$
\begin{aligned}
W_\alpha &= \frac{i}{2}[\overline{\nabla}^{\dot\alpha},\{\nabla_\alpha,\overline{\nabla}_{\dot\alpha}\}] = i\overline{D}^2(e^{-V}D_\alpha e^V)\,, \\
W_{\dot\alpha} &= \frac{i}{2}[\nabla^{\alpha},\{\overline{\nabla}_{\dot\alpha},\nabla_\alpha\}] = e^{-V}iD^2(e^V\overline{D}_{\dot\alpha}e^{-V})e^V\,, \\
F_{\alpha z} &= i[\nabla_\alpha,\nabla_z] = D_\alpha\Omega - i\partial(e^{-V}D_\alpha e^V) + [(e^{-V}D_\alpha e^V),\Omega]\,, \\
F_{\dot\alpha\overline{z}} &= i[\overline{\nabla}_{\dot\alpha},\overline{\nabla}_{\overline{z}}] = i\overline{D}_{\dot\alpha}(e^{-V}\overline{\partial}e^V) + \overline{D}_{\dot\alpha}(e^{-V}\overline{\Omega}e^V)\,, \\
F_{z\overline{z}} &= i[\nabla_z,\overline{\nabla}_{\overline{z}}] = i\partial(e^{-V}\overline{\partial}e^V) + \partial(e^{-V}\overline{\Omega}e^V) - \overline{\partial}\Omega + \\
&\quad + [\Omega,(e^{-V}\overline{\partial}e^V)] - i[\Omega,(e^{-V}\overline{\Omega}e^V)]\,.
\end{aligned}
\tag{3.4}
$$

The gauge invariant action in six dimensions is ($d^{10}Z \equiv d^6x d^4\theta$, $d^8Z \equiv d^6x d^2\theta$)

$$
\begin{aligned}
S_{SYM_6}[V,\Omega,\overline{\Omega}] = \frac{1}{2g^2}\mathrm{Tr}\int d^8Z\; W^\alpha W_\alpha + \\
+\frac{1}{g^2}\mathrm{Tr}\int d^{10}Z\Bigg[e^{-V}\overline{\Omega}e^V\Omega + i(\partial e^{-V})\overline{\Omega}e^V - ie^V\Omega(\overline{\partial}e^{-V}) \\
+\frac{1}{2}(e^{-V}\overline{\partial}e^V)(e^{-V}\partial e^V) + (\overline{\partial}V)\left(\frac{\sinh L_V - L_V}{(L_V)^2}\right)(\partial V)\Bigg],
\end{aligned}
\tag{3.5}
$$

where g is the gauge coupling constant of dimension -1. The equations of motion from its variation are

$$
\{\nabla^\alpha, W_\alpha\} - \frac{1}{2}F_{z\overline{z}} = 0\,, \qquad \{\nabla^\alpha, F_{\alpha z}\} = 0\,. \tag{3.6}
$$

When reduced to components with the auxiliary fields integrated out, this action describes the dynamics of a 6D, $\mathcal{N}=(0,1)$ vector multiplet given by a 6D vector field and a $(0,1)$ Weyl spinor [12, 3]. The real superfield V contains the 4D part of the 6D vector field and half of the 6D Weyl spinor. The connections Ω and $\overline{\Omega}$ contain the rest of the physical degrees of freedom [3, 12].

By dimensional reduction, the previous action can be derived from the ten dimensional $\mathcal{N}=1$ supersymmetric action found in [12]. Proceeding in this way what one finds is a 6D $\mathcal{N}=(1,1)$ SYM, where the vector multiplet is described by the action (3.5) and it is minimally coupled to a CC hypermultiplet in the adjoint representation of the gauge group. Setting to zero the hypermultiplet we are left with the action for the $(0,1)$ vector multiplet.

As for the hypermultiplets, in order to describe a vector multiplet with opposite chirality it is sufficient to exchange $\partial \leftrightarrow -\overline{\partial}$ in the previous formulation. In particular, for the $\mathcal{N}=(1,0)$ SYM we impose the constraints

$$
\begin{aligned}
F_{\alpha\beta} &= i\{\nabla_\alpha,\nabla_\beta\} = 0\,, & F_{\dot\alpha\dot\beta} &= i\{\overline{\nabla}_{\dot\alpha},\overline{\nabla}_{\dot\beta}\} = 0\,, \\
F_{\alpha z} &= i[\nabla_\alpha,\nabla_z] = 0\,, & F_{\dot\alpha\overline{z}} &= i[\overline{\nabla}_{\dot\alpha},\overline{\nabla}_{\overline{z}}] = 0\,,
\end{aligned}
\tag{3.7}
$$

whereas $F_{\alpha\overline{z}}$, $F_{\dot\alpha z}$ are non-trivial. In the chiral representation for the 4D superspace covariant derivatives, these constraints imply the definitions (3.2) are modified as long as $(z,\overline{z})$-derivatives are concerned, according to

$$
\nabla_z = \partial - i\Gamma_z = e^{-V}(\partial - i\overline{\Omega})e^V\,, \qquad \overline{\nabla}_{\overline{z}} = \overline{\partial} - i\overline{\Gamma}_{\overline{z}} = \overline{\partial} - i\Omega\,, \tag{3.8}
$$

Therefore, in the $\mathcal{N}=(1,0)$ case the chiral connection is $\overline{\Gamma}_{\overline{z}} = \Omega$.

Now we study the minimal coupling of hypermultiplets to 6D SYM. This can be accomplished by simply changing the definitions of the hypermultiplets to insure covariance under gauge transformations. Similarly to the 4D, $\mathcal{N}=1$ case, this amounts to implement all the derivatives in the constraints and in the actions to be the gauge covariant derivatives in eq. (3.2).

The coupling of the 6D CC hypermultiplet to SYM has been given in [3]. Following the same procedure it is easy to couple the hypermultiplet when it is formulated in terms of CNM superfields. Therefore, we treat the two cases together.

We consider $\mathcal{N}=(1,0)$ covariantly CC and CNM hypermultiplets belonging to a given representation of the gauge group and defined by the following covariant constraints

$$\overline{\nabla}_{\dot\alpha}\Omega_{c\pm}=0\,,\qquad \nabla_\alpha\overline{\Omega}_{c\pm}=0\,, \tag{3.9}$$

$$\overline{\nabla}_{\dot\alpha}\Phi_c=0\,,\qquad \nabla_\alpha\overline{\Phi}_c=0\,,\qquad \overline{\nabla}^2\Sigma_c=\nabla_z\Phi_c\,,\qquad \nabla^2\overline{\Sigma}_c=\overline{\nabla}_{\overline{z}}\overline{\Phi}_c\,. \tag{3.10}$$

The corresponding gauge invariant actions read

$$S_{CC}=\int d^{10}Z\Big[\,\overline{\Omega}_{c+}\Omega_{c+}+\overline{\Omega}_{c-}\Omega_{c-}\Big]+\int d^8Z\Big[\Omega_{c+}\nabla_z\,\Omega_{c-}\Big]+\int d^8\overline{Z}\Big[\overline{\Omega}_{c+}\overline{\nabla}_{\overline{z}}\overline{\Omega}_{c-}\Big]\,, \tag{3.11}$$

$$S_{CNM}=\int d^{10}Z\Big[\,\overline{\Phi}_c\Phi_c-\overline{\Sigma}_c\Sigma_c\,\Big]\,. \tag{3.12}$$

As an interesting point, we note the constraints (3.3) on the covariant derivatives follow as consistency conditions for the existence of $\mathcal{N}=(1,0)$ covariant CNM hypermultiplets (3.10). Therefore, $\mathcal{N}=(1,0)$ CNM hypermultiplets can only be coupled to $\mathcal{N}=(0,1)$ SYM vector multiplets. This translates into a pure kinematic language, the well-known fact in six dimensions, hypermultiplets with a given chirality can only be coupled to vector multiplets of opposite chirality [3, 11]. In the case of the CC formulation of hypermultiplets the same condition arises at the dynamical level, since the action as written in eq. (3.11) makes sense only if the derivative ∇_z ($\overline{\nabla}_{\overline{z}}$) does not spoil the chirality of the superfield $\Omega_{c\pm}$ ($\overline{\Omega}_{c\pm}$).

In the particular case of matter in the adjoint representation of the gauge group we find it convenient to re-express the actions in terms of ordinary (non-covariant) superfields. Therefore, given chiral superfields $\Omega_\pm$ and Φ satisfying $\overline{D}_{\dot\alpha}\Omega_\pm=\overline{D}_{\dot\alpha}\Phi=0$ and a complex-linear multiplet with modified conditions

$$\overline{D}^2\Sigma=\partial\Phi-i[\Omega,\Phi]\,,\qquad D^2\overline{\Sigma}=\overline{\partial}\,\overline{\Phi}-i[\overline{\Omega},\overline{\Phi}]\,. \tag{3.13}$$

the previous actions can be re-written as

(CC)

$$\begin{aligned}&\mathrm{Tr}\int d^{10}Z\Big[e^{-V}\overline{\Omega}_+e^V\Omega_++e^{-V}\overline{\Omega}_-e^V\Omega_-\Big]+\\&+\left\{\mathrm{Tr}\int d^8Z\Big[\Omega_+\partial\,\Omega_--i\,\Omega_+[\Omega,\Omega_-]\Big]+(\mathrm{h.c.})\right\}\,,\end{aligned} \tag{3.14}$$

(CNM)

$$\mathrm{Tr}\int d^{10}Z\Big[\,e^{-V}\overline{\Phi}e^V\Phi-e^{-V}\overline{\Sigma}e^V\Sigma\,\Big]\,, \tag{3.15}$$

respectively. Using a gauge covariant generalization of the duality transformations described in section 2, it is straightforward to prove the two theories are still dual.

Adding the action (3.5) for the $\mathcal{N}=(0,1)$ vector multiplet ($g=1$ for simplicity) to (3.14) we have the action describing 6D, $\mathcal{N}=(1,1)$ SYM as obtained by dimensional reduction of the 10D $\mathcal{N}=1$ SYM of [12]. Instead, if we add (3.5) to (3.15) we find an action for a non-minimal representation of the 6D $\mathcal{N}=(1,1)$ SYM. Both the resulting theories, when dimensionally reduced to four dimensions, give the on-shell $\mathcal{N}=4$ SYM, in the second case in a non-minimal representation.

4. Quantization of the 6D multiplets

In this section we perform the quantization of the 6D multiplets considered in the previous sections, using a 4D, $\mathcal{N}=1$ covariant procedure. We concentrate only on multiplets of a given 6D chirality, the cases with opposite chirality being completely analogue.

CC hypermultiplet. Given the $\mathcal{N}=(1,0)$ hypermultiplet in the CC formulation, we proceed to the quantization of its action (2.3). In complete analogy with the 4D, $\mathcal{N}=1$ case [13], we first solve the (anti)chiral constraints (2.2) by expressing $\Omega_\pm$ ($\overline{\Omega}_\pm$) in terms of two unconstrained superfields

$$\Omega_\pm = \overline{D}^2\psi_\pm\,, \qquad \overline{\Omega}_\pm = D^2\overline{\psi}_\pm\,. \tag{4.1}$$

The kinetic action (2.3) becomes

$$S^0_{CC} = \int d^{10}Z\ (\ \overline{\psi}_-\ ,\ \psi_+\ ) \begin{pmatrix} D^2\overline{D}^2 & -\overline{\partial}D^2 \\ \partial\overline{D}^2 & \overline{D}^2D^2 \end{pmatrix} \begin{pmatrix} \psi_- \\ \overline{\psi}_+ \end{pmatrix}\ . \tag{4.2}$$

As a consequence of the invariance of the action under $\delta\psi_\pm = \overline{D}^{\dot\alpha}\overline{\chi}_{\pm\dot\alpha}$ ($\delta\overline{\psi}_\pm = D^\alpha\chi_{\pm\alpha}$), the kinetic operator in (4.2) is not invertible. We can fix these invariances using the well known gauge-fixing procedure for the four dimensional massless scalar chiral superfield [13]. This amounts to add gauge-fixing terms which complete the operators $D^2\overline{D}^2$ and $\overline{D}^2D^2$ to $\Box_4$ (see [13] for details). The kinetic action (4.2) then reads

$$S^0_{CC} + S_{GF} = \int d^{10}Z\ (\ \overline{\psi}_-\ ,\ \psi_+\ ) \begin{pmatrix} \Box_4 & -\overline{\partial}D^2 \\ \partial\overline{D}^2 & \Box_4 \end{pmatrix} \begin{pmatrix} \psi_- \\ \overline{\psi}_+ \end{pmatrix}\ . \tag{4.3}$$

Moreover, as in the ordinary 4D case, at the end of the gauge-fixing procedure the ghosts decouple from the physical superfields. The kinetic operator in (4.3) is now invertible and from its inverse we find the following propagators

$$\begin{pmatrix} \langle\psi_-\overline{\psi}_-\rangle & \langle\psi_-\psi_+\rangle \\ \langle\overline{\psi}_+\overline{\psi}_-\rangle & \langle\overline{\psi}_+\psi_+\rangle \end{pmatrix} = \frac{1}{\Box_4} \begin{pmatrix} \left(\frac{D^2\overline{D}^2}{\Box_4}\frac{\partial\overline{\partial}}{\Box_6} - 1\right) & -\frac{\partial D^2}{\Box_6} \\ \frac{\overline{\partial}\overline{D}^2}{\Box_6} & \left(\frac{\overline{D}^2D^2}{\Box_4}\frac{\partial\overline{\partial}}{\Box_6} - 1\right) \end{pmatrix}\ . \tag{4.4}$$

Using the definitions (4.1), we finally have the propagators for the physical superfields

$$\begin{aligned} \langle\Omega_-\overline{\Omega}_-\rangle &= -\frac{\overline{D}^2D^2}{\Box_6}\delta^{10}(Z-Z')\,, & \langle\Omega_-\Omega_+\rangle &= -\frac{\partial\overline{D}^2}{\Box_6}\delta^{10}(Z-Z')\,, \\ \langle\overline{\Omega}_+\overline{\Omega}_-\rangle &= \frac{\overline{\partial}D^2}{\Box_6}\delta^{10}(Z-Z')\,, & \langle\overline{\Omega}_+\Omega_+\rangle &= -\frac{D^2\overline{D}^2}{\Box_6}\delta^{10}(Z-Z')\ . \end{aligned} \tag{4.5}$$

CNM Hypermultiplet. In order to perform the quantization of the CNM hypermultiplet we start from the action (2.8) for Φ chiral and Σ satisfying the more general constraints

$$\overline{D}^2\Sigma = \partial\Phi + \Phi P(\Phi,\Omega_a)\,, \qquad D^2\overline{\Sigma} = \overline{\partial}\,\overline{\Phi} + \overline{\Phi}\,\overline{P}(\overline{\Phi},\overline{\Omega}_{\overline{a}})\ , \tag{4.6}$$

where Ω_a ($\overline{\Omega}_{\overline{a}}$) are a set of (anti)chiral superfields and P ($\overline{P}$) is a (anti)holomorphic function, analytic in the superfields, with an expansion which starts from the linear order. The constraints (2.6) and (3.13) are particular cases of (4.6), with $P = 0$ and $\Phi P(\Phi,\Omega) = i[\Phi,\Omega]$, respectively.

In order to quantize the action (2.8), we follow closely the procedure used in [18] for the 4D case. First of all we solve the kinematical constraints (4.6) which define Φ and Σ. The most general solution is given in terms of unconstrained superfields as

$$\Phi \equiv \overline{D}^2\chi\,, \qquad \Sigma = \overline{D}^{\dot\alpha}\overline{\sigma}_{\dot\alpha} + \partial\chi + \chi P(\Phi,\Phi_a)\,, \tag{4.7}$$

$$\overline{\Phi} \equiv D^2\overline{\chi}\,, \qquad \overline{\Sigma} = D^\alpha\sigma_\alpha + \overline{\partial}\,\overline{\chi} + \overline{\chi}\overline{P}(\overline{\Phi},\overline{\Phi}_{\overline{a}})\,. \tag{4.8}$$

The action (2.8) then reads

$$S_{CNM} = \int d^{10}Z\Big[(D^2\overline{\chi})(\overline{D}^2\chi) - \Big(D^\alpha\sigma_\alpha + \overline{\partial}\,\overline{\chi} + \overline{\chi}\overline{P}\Big)\Big(\overline{D}^{\dot\alpha}\overline{\sigma}_{\dot\alpha} + \partial\chi + \chi P\Big)\Big]\,, \tag{4.9}$$

whose quadratic part is

$$S^0_{CNM} = \int d^{10}Z\ (\ \overline{\chi}\ ,\ \sigma_\alpha\ )\begin{pmatrix} (D^2\overline{D}^2 + \overline{\partial}\partial) & \overline{\partial}\,\overline{D}^{\dot\alpha} \\ -\partial D^\alpha & -D^\alpha\overline{D}^{\dot\alpha}\end{pmatrix}\begin{pmatrix}\chi\\ \overline{\sigma}_{\dot\alpha}\end{pmatrix}\ . \tag{4.10}$$

The expressions (4.7), (4.8) and the action (4.9) are invariant under the following two sets of tranformations

$$\delta\chi = \overline{D}^{\dot\alpha}\overline{\chi}_{\dot\alpha}\,, \qquad \delta\overline{\sigma}_{\dot\alpha} = -\partial\,\overline{\chi}_{\dot\alpha} - \overline{\chi}_{\dot\alpha}P(\Phi,\Phi_a), \tag{4.11}$$

and

$$\begin{aligned}
\delta\chi &= 0\ , \\
\delta\sigma^\alpha &= D_\beta\sigma^{(\beta\alpha)}\ , \\
\delta\sigma^{(\beta\alpha)} &= D_\gamma\sigma^{(\gamma\beta\alpha)}\ , \\
\delta\sigma^{(\gamma\beta\alpha)} &= D_\delta\sigma^{(\delta\gamma\beta\alpha)}\ , \\
&\vdots \\
\delta\sigma^{(\alpha_n\alpha_{n-1}\cdots\alpha_1)} &= D_{\alpha_{n+1}}\sigma^{(\alpha_{n+1}\alpha_n\alpha_{n-1}\cdots\alpha_1)}\ , \\
&\vdots
\end{aligned} \tag{4.12}$$

Therefore, the kinetic operator in (4.10) is not invertible.

Since these invariances are due to the four-dimensional superspace structure of the covariant derivatives, we can apply the gauge-fixing procedure of [18] forgetting we are working in six dimensions.

The gauge-fixing procedure runs in two steps. First, we consider the transformations (4.11). As for the CC hypermultiplet, to fix this invariance we use the standard gauge-fixing procedure of the four dimensional massless scalar chiral superfield [13] to bring the kinetic operator $(D^2\overline{D}^2 + \overline{\partial}\partial)$ in (4.10) to $(\Box_4 + \overline{\partial}\partial) = \Box_6$.

As a second step we consider the transformations (4.12). Since χ and $\overline{\chi}$ do not transform, we can use the gauge-fixing procedure of [19] for the ordinary four dimensional complex-linear superfield and modify only the σ_α, $\overline{\sigma}_{\dot\alpha}$ part of the kinetic action. Precisely, the gauge-fixing is performed by introducing an infinite tower of ghosts according to a non-trivial superspace version of the Batalin-Vilkovinsky formalism. As proved in [19], if the gauge-fixing functions are chosen to be independent of the background physical fields, the tower of ghosts can be completely decoupled by a finite number of ghost fields redefinitions, and we are left with an invertible kinetic term for σ_α, $\overline{\sigma}_{\dot\alpha}$.

The same procedure can be applied without modifications to our case. The result is the conversion of the operator $D^\alpha\overline{D}^{\dot\alpha}$ into the invertible operator

$$W^{\alpha\dot\alpha} = \left[D^\alpha\overline{D}^{\dot\alpha} + \frac{k^2}{2}\overline{D}^{\dot\alpha}D^\alpha - \frac{k^2}{2}\left(\frac{1+k_1'^{\,2}}{1-k_1'^{\,2}}\right) i\partial^{\alpha\dot\alpha}\frac{\overline{D}^2D^2}{\Box_4} + \frac{k^2}{2}i\partial^{\alpha\dot\alpha}\frac{D_\beta\overline{D}^2D^\beta}{\Box_4}\right], \quad (4.13)$$

where k and k_1' are two gauge-fixing parameters.

At the end of the procedure the gauge-fixed action reads

$$S_{CNM} + S^{tot}_{GF} = \int d^{10}Z\ (\ \overline{\chi}\ ,\ \sigma_\alpha\ ) \begin{pmatrix} \Box_6 & \overline{\partial}\,\overline{D}^{\dot\alpha} \\ -\partial D^\alpha & -W^{\alpha\dot\alpha} \end{pmatrix} \begin{pmatrix} \chi \\ \overline{\sigma}_{\dot\alpha} \end{pmatrix}. \quad (4.14)$$

Inverting the kinetic operator (4.14) we find

$$\begin{pmatrix} \langle\chi\overline{\chi}\rangle & \langle\chi\sigma_\alpha\rangle \\ \langle\overline{\sigma}_{\dot\alpha}\overline{\chi}\rangle & \langle\overline{\sigma}_{\dot\alpha}\sigma_\alpha\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\Box_4}\left(\frac{\overline{\partial}\partial}{\Box_6}\frac{D^2\overline{D}^2}{\Box_4} - 1\right) & \frac{\overline{\partial}}{\Box_4}\left(\frac{1}{2}\frac{\overline{D}^2D_\alpha}{\Box_4} - \frac{D_\alpha\overline{D}^2}{\Box_4}\right) \\ \frac{\partial}{\Box_4}\left(\frac{1}{2}\frac{\overline{D}_{\dot\alpha}D^2}{\Box_4} - \frac{D^2\overline{D}_{\dot\alpha}}{\Box_4}\right) & W^{-1}_{\alpha\dot\alpha} + \frac{\overline{\partial}\partial}{\Box_4}W^{-1}_{\beta\dot\alpha}D^\beta\overline{D}^{\dot\beta}W^{-1}_{\alpha\dot\beta} \end{pmatrix}, \quad (4.15)$$

where

$$\begin{aligned} W^{-1}_{\alpha\dot\alpha} =& -\frac{i\partial_{\alpha\dot\alpha}}{\Box_4} + \frac{3(kk_1')^2 + 4 - 2k_1'^{\,2}}{4(kk_1')^2}i\partial_{\alpha\dot\alpha}\frac{\overline{D}^2D^2}{\Box_4^2} + \frac{3k^2-2}{4k^2}i\partial_{\alpha\dot\alpha}\frac{D_\beta\overline{D}^2D^\beta}{\Box_4^2} + \\ &+ \frac{2-k^2}{4k^2}i\partial_{\alpha\dot\beta}i\partial_{\beta\dot\alpha}\frac{\overline{D}^{\dot\beta}D^\beta}{\Box_4^2}, \end{aligned} \quad (4.16)$$

is the inverse of $W^{\alpha\dot\alpha}$.

In the particular case of CNM multiplet in (2.6), $P = \overline{P} \equiv 0$, we easily infer the propagators of the physical superfields

$$\begin{aligned} \langle\Phi\overline{\Phi}\rangle &= -\frac{\overline{D}^2D^2}{\Box_6}\delta^{10}(Z-Z'), \qquad \langle\Sigma\overline{\Phi}\rangle = -\frac{\partial D^2}{\Box_6}\delta^{10}(Z-Z'), \\ \langle\Phi\overline{\Sigma}\rangle &= \frac{\overline{\partial}\,\overline{D}^2}{\Box_6}\delta^{10}(Z-Z'), \qquad \langle\Sigma\overline{\Sigma}\rangle = \left[1 - \frac{D^2\overline{D}^2}{\Box_6}\right]\delta^{10}(Z-Z'). \end{aligned} \quad (4.17)$$

Vector multiplet. The quantization of the $\mathcal{N}=(0,1)$ vector multiplet can be performed by following closely the procedure described in [12] for the 10D case. For simplicity we set $g=1$ in (3.5).

The quadratic part of the action (3.5) is

$$S^{(2)}_{SYM_6}[V,\Omega,\overline{\Omega}] = \mathrm{Tr}\int d^{10}Z\Big[-\frac{1}{2}VD_\alpha\overline{D}^2D^\alpha V+\overline{\Omega}\Omega-i(\partial V)\overline{\Omega}+i\Omega(\overline{\partial}V)+\frac{1}{2}(\overline{\partial}V)(\partial V)\Big]\,. \tag{4.18}$$

invariant under the gauge transformations (3.1). To fix this invariance we choose a Feynman-type gauge-fixing term [12] suitably adapted to the six dimensional case

$$S_{GF} = -\mathrm{Tr}\int d^{10}Z\left(\overline{D}^2V+i\,\frac{\overline{D}^2}{\Box_4}\,\partial\,\overline{\Omega}\right)\left(D^2V-i\,\frac{D^2}{\Box_4}\,\overline{\partial}\,\Omega\right)\,. \tag{4.19}$$

The corresponding Faddev-Popov ghosts action is

$$\begin{aligned} S_{FP} = -\mathrm{Tr}\int d^{10}Z\Bigg[&(c'+\overline{c}')L_{\frac{V}{2}}\Big((c+\overline{c})+\coth L_{\frac{V}{2}}(c-\overline{c})\Big)-c'\,\frac{\partial\overline{\partial}}{\Box_4}\,\overline{c}\\ &-i(\partial c')\frac{1}{\Box_4}[\overline{\Omega},\overline{c}]+\overline{c}'\,\frac{\partial\overline{\partial}}{\Box_4}\,c+i(\overline{\partial}\,\overline{c}')\frac{1}{\Box_4}[\Omega,c]\Bigg]\,. \end{aligned} \tag{4.20}$$

The advantage of using the gauge-fixing term (4.19) is in the quadratic part of the action the superfield V decouples from Ω and $\overline{\Omega}$

$$S^{(2)} = \mathrm{Tr}\int d^{10}Z\left[-\frac{1}{2}V\Box_6V+\overline{\Omega}\,\frac{\Box_6}{\Box_4}\,\Omega+\mathrm{Ghosts}\right]\,. \tag{4.21}$$

and the propagators in the chosen gauge are

$$\langle VV\rangle = \frac{1}{\Box_6}\delta^{10}(Z-Z')\,,\qquad \langle\overline{\Omega}\Omega\rangle = -\frac{D^2\overline{D}^2}{\Box_6}\delta^{10}(Z-Z')\,. \tag{4.22}$$

We conclude this section with few comments. Given the particular approach we have used to study six dimensional superfields, the quantization turns out to be not much affected by working in six dimensions and the results are very similar to the four-dimensional case. In particular, a formal equivalence between 4D CC/CNM massive propagators and 6D CC/CNM massless ones can be established by identifying the 4D complex mass with the extra dimensions derivative operators, $m\leftrightarrow\partial$, $\overline{m}\leftrightarrow-\overline{\partial}$. In the case of the vector multiplet the corrispondence would work with a 4D massive vector multiplet written in a superspace Stueckelberg formalism [20]. The previous correspondence could be very useful when studying quantum properties of six dimensional theories in a 4D $\mathcal{N}=1$ formalism.

5. The 6D projective superspace perspective

In four dimensions, the complex-linear superfield plays an important role in the definition of $\mathcal{N}=2$ multiplets in the context of projective superspace [5, 4, 6, 21]. In fact, the on-shell $\mathcal{N}=1$ superspace description of the $\mathcal{N}=2$ (ant)artic projective superfield is given in

terms of a 4D CNM multiplet [5, 4, 6, 21]. Having constructed 6D hypermultiplets using a CNM multiplet, it is then natural to ask if a projective superspace formulation of 6D supersymmetry exits and if one can define there 6D superfields whose on-shell version is given by the multiplets previously introduced.

Since we have defined 6D multiplets always keeping manifest only the 4D superspace structures, we try to formulate the 6D projective superspace using the same approach.[2] In this way we take advantage of the fact that the 4D projective superspace manifestly preserves many structures of the ordinary $\mathcal{N}=1$ superspace.

First of all we remind our readers the algebra of the $\mathcal{N}=(1,0)$ supercovariant derivatives is

$$\{D^{a\tilde{\alpha}}, D^{b\tilde{\beta}}\} = \epsilon^{ab}\Gamma^{\mu\tilde{\alpha}\tilde{\beta}} i\partial_\mu \,, \tag{5.1}$$

where $\Gamma^{\mu\tilde{\alpha}\tilde{\beta}}$ have been defined in the appendix, ϵ^{ab} is the invariant tensor of the SU(2) automorphism group of the $\mathcal{N}=(1,0)$ algebra and the derivatives $D^{a\tilde{\alpha}}$ are $(1,0)$ Weyl spinors satisfying a SU(2)-Majorana condition [11]. For our purposes the algebra (5.1) can be written using the 4D spinor notation as

$$\begin{aligned} &\{D_{a\alpha}, D_{b\beta}\} = \epsilon_{ab} C_{\alpha\beta}\overline{\partial}\,, \qquad \{\overline{D}^a_{\dot\alpha}, \overline{D}^b_{\dot\beta}\} = \epsilon^{ab} C_{\dot\alpha\dot\beta}\partial\,, \\ &\{D_{a\alpha}, \overline{D}^b_{\dot\beta}\} = \delta^b_a i\partial_{\alpha\dot\beta}\,. \end{aligned} \tag{5.2}$$

The interesting point is this algebra has the same structure of the 4D $\mathcal{N}=2$ algebra with a complex central charge given by $(\epsilon_{ab}\overline{\partial})$. Therefore, we can generalize to six dimensions the construction of the projective superspace for the case of an underlying 4D $\mathcal{N}=2$ SUSY with central charge, as given in [5].

We parametrize the projective superspace with a complex coordinate ζ and we define the projective supercovariant derivatives as

$$\nabla_\alpha(\zeta) = \zeta D_{1\alpha} - D_{2\alpha}\ ,\ \overline{\nabla}_{\dot\alpha}(\zeta) = \overline{D}^1_{\dot\alpha} + \zeta\overline{D}^2_{\dot\alpha}\,, \tag{5.3}$$

with the orthogonal set of supercovariant derivatives given by

$$\Delta_\alpha(\zeta) = D_{1\alpha} + \frac{1}{\zeta}D_{2\alpha}\ ,\ \overline{\Delta}_{\dot\alpha}(\zeta) = \overline{D}^2_{\dot\alpha} - \frac{1}{\zeta}\overline{D}^1_{\dot\alpha}\,. \tag{5.4}$$

The projective supercovariant derivatives algebra is

$$\begin{aligned} \{\nabla_\alpha(\zeta), \nabla_\beta(\zeta)\} &= \{\nabla_\alpha(\zeta), \overline{\nabla}_{\dot\alpha}(\zeta)\} = \{\overline{\nabla}_{\dot\alpha}(\zeta), \overline{\nabla}_{\dot\beta}(\zeta)\} = 0\,, \\ \{\Delta_\alpha(\zeta), \Delta_\beta(\zeta)\} &= \{\Delta_\alpha(\zeta), \overline{\Delta}_{\dot\alpha}(\zeta)\} = \{\overline{\Delta}_{\dot\alpha}(\zeta), \overline{\Delta}_{\dot\beta}(\zeta)\} = 0\,, \\ \{\nabla_\alpha(\zeta), \Delta_\beta(\zeta)\} &= 2C_{\alpha\beta}\overline{\partial}\,, \qquad \{\overline{\nabla}_{\dot\alpha}(\zeta), \overline{\Delta}_{\dot\beta}(\zeta)\} = 2C_{\dot\alpha\dot\beta}\partial\,, \\ \{\nabla_\alpha(\zeta), \overline{\Delta}_{\dot\alpha}(\zeta)\} &= -\{\overline{\nabla}_{\dot\alpha}(\zeta), \Delta_\alpha(\zeta)\} = -2i\partial_{\alpha\dot\alpha}\,. \end{aligned} \tag{5.5}$$

Following [5, 4, 6, 21], superfields living on the projective superspace are constrained by

$$\nabla_\alpha(\zeta)\Xi = 0 = \overline{\nabla}_{\dot\alpha}(\zeta)\Xi \qquad \Longrightarrow \qquad D_{2\alpha}\Xi = \zeta D_{1\alpha}\Xi\ ,\ \overline{D}^1_{\dot\alpha}\Xi = -\zeta\overline{D}^2_{\dot\alpha}\Xi\ . \tag{5.6}$$

[2] Recently, in [16] a similar extension has been found for the five dimensional case. Note also that a previous formulation of 6D projective superspace was provided in [17] to describe the $O(2)$ tensor multiplet in six dimensions.

Now, the projective superfield Ξ is a function of the six bosonic coordinates, of the grassmannian $(\theta^{a\alpha}, \overline{\theta}_a^{\dot{\alpha}})$ and it is chosen to be holomorphic in ζ on $\mathbb{C}^*$. It can be expanded as

$$\Xi(x_i, \theta^{a\alpha}, \overline{\theta}_a^{\dot{\alpha}}, \zeta) = \sum_{n=-\infty}^{+\infty} \Xi_n(x_i, \theta^{a\alpha}, \overline{\theta}_a^{\dot{\alpha}})\, \zeta^n \,, \tag{5.7}$$

where Ξ_n are $\mathcal{N} = 2$ superfields satisfying

$$D_{2\alpha}\Xi_{n+1} = D_{1\alpha}\Xi_n \,, \qquad \overline{D}^2_{\dot{\alpha}}\Xi_n = -\overline{D}^1_{\dot{\alpha}}\Xi_{n+1} \,, \tag{5.8}$$

as follows from eqs. (5.6). The above constraints fix the dependence of the Ξ_n on half of the Grassmannian coordinates $(\theta^{a\alpha}, \overline{\theta}_a^{\dot{\alpha}})$ of the superspace. Therefore, Ξ_n can be considered as superfields which effectively live on a $\mathcal{N} = 1$ superspace with for example $\theta^\alpha = \theta^{1\alpha}$, $\overline{\theta}^{\dot{\alpha}} = \overline{\theta}_1^{\dot{\alpha}}$ [5, 4, 6, 21].

In projective superspace the natural conjugation operation combines complex conjugation with the antipodal map on the Riemann sphere ($\zeta \to -1/\zeta$) and acts on projective superfields as

$$\breve{\Xi} = \sum_{n=-\infty}^{+\infty} \breve{\Xi}_n\, \zeta^n = \sum_{n=-\infty}^{+\infty} (-1)^n \overline{\Xi}_{-n}\, \zeta^n \,. \tag{5.9}$$

Similarly to the 4D case [5, 4, 6, 21], six dimensional $\mathcal{N} = (1,0)$ SUSY invariant actions have then the general form

$$\int d^6x \left\{ \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta} D^2\overline{D}^2 \mathcal{L}(\Xi, \breve{\Xi}, \zeta)\Big| \right\} \,, \tag{5.10}$$

where $\mathcal{L}(\Xi, \breve{\Xi}, \zeta)$ is real under the $\smile$-conjugation (5.9) and C is a contour around the origin of the complex ζ-plane.

We have constructed a projective superspace which seems to have the right properties to generalize the $\mathcal{N} = 1$ formalism used in the previous sections to study 6D supersymmetric models. The non-trivial property of this formulation is the linear realization of the 6D, $\mathcal{N} = (1,0)$ supersymmetry. Now we construct 6D projective superspace multiplets which have the same physical content as the CNM multiplet and the vector multiplet previously considered.

Polar Hypermultiplet. We study the polar hypermultiplet described by (ant)artic superfields. In 4D it is the natural generalization of the CNM multiplet. We define the artic and antartic projective superfields as

$$\Upsilon = \sum_{n=0}^{+\infty} \Upsilon_n\, \zeta^n \;,\; \breve{\Upsilon} = \sum_{n=0}^{+\infty} (-1)^n \overline{\Upsilon}_n \frac{1}{\zeta^n} \quad . \tag{5.11}$$

Due to the truncation of the series, the $\mathcal{N} = 1$ constraints on the component superfields Υ_n are

$$\begin{array}{ll} \overline{D}_{\dot{\alpha}}\Upsilon_0 = 0 \,, & \overline{D}^2\Upsilon_1 = \partial\,\Upsilon_0 \,, \\ D_\alpha\overline{\Upsilon}_0 = 0 \,, & D^2\overline{\Upsilon}_1 = \overline{\partial}\,\overline{\Upsilon}_0 \,, \end{array} \tag{5.12}$$

with Υ_n ($\overline{\Upsilon}_n$) $n > 1$ unconstrained $\mathcal{N} = 1$ superfields. The natural action for a free polar multiplet is then

$$\int d^6x d^4\theta \left\{ \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta} \breve{\Upsilon} \Upsilon \right\} = \int d^6x d^4\theta \left\{ \sum_{n=0}^{+\infty} (-1)^n \overline{\Upsilon}_n \Upsilon_n \right\} . \tag{5.13}$$

The polar multiplet describes a generalization of the 6D $\mathcal{N} = (1,0)$ CNM hypermultiplet introduced in section 2, once we identify $\Phi \equiv \Upsilon_0$ ($\overline{\Phi} \equiv \overline{\Upsilon}_0$) and $\Sigma \equiv \Upsilon_1$ ($\overline{\Sigma} \equiv \overline{\Upsilon}_1$) in complete analogy with the four dimensional case. The infinite set of auxiliary superfields in the polar multiplet extend the CNM hypermultiplet to a multiplet which transforms linearly under 6D $\mathcal{N} = (1,0)$ supersymmetry.

Tropical Multiplet. Another interesting multiplet to consider in the framework of projective superspace is the real tropical multiplet [5, 4, 6]. This is defined in terms of a projective superfield $V(\zeta, \overline{\zeta})$ which is analytic away from the poles of the Riemann sphere and real under the $\smile$-conjugation. Therefore, its expansion reads

$$V = \sum_{-\infty}^{+\infty} V_n \, \zeta^n \ , \ V_{-n} = (-1)^n \overline{V}_n \, . \tag{5.14}$$

In four dimensions, the real tropical multiplet is the prepotential of $\mathcal{N} = 2$ SYM. In the abelian case the explicit expression for the action is known[3] [5, 4, 6].

Inspired by the 4D case, we consider the following action for a real tropical projective superfield

$$S_{PSYM_6} = -\frac{1}{2} \int d^6x d^8\theta \oint \frac{d\zeta_1}{2\pi i} \frac{d\zeta_2}{2\pi i} \frac{V(\zeta_1) V(\zeta_2)}{(\zeta_1 - \zeta_2)^2} \, . \tag{5.15}$$

This is formally equal to the action for a free real tropical projective superfield in 4D [5, 4, 6] with the only difference now that the fields live in a 6D superspace. Now, using the general constraints (5.6) together with the identities

$$Q^2 \Xi = \zeta^2 D^2 \Xi + \zeta \overline{\partial} \, \Xi \, , \qquad \overline{Q}^2 \Xi = \frac{1}{\zeta^2} \overline{D}^2 \Xi - \frac{1}{\zeta} \partial \, \Xi \, , \tag{5.16}$$

where we have defined $D_\alpha = D_{1\alpha}$, $\overline{D}_{\dot{\alpha}} = \overline{D}^1_{\dot{\alpha}}$, $Q_\alpha = D_{2\alpha}$, $\overline{Q}_{\dot{\alpha}} = \overline{D}^2_{\dot{\alpha}}$, from (5.15) we find

$$\begin{aligned} S_{PSYM_6} &= -\frac{1}{2} \int d^6x D^2 \overline{D}^2 \oint \frac{d\zeta_1}{2\pi i} \frac{d\zeta_2}{2\pi i} \frac{1}{(\zeta_1 - \zeta_2)^2} \Big(Q^2 \overline{Q}^2 V(\zeta_1) V(\zeta_2) \Big) \Big| \\ &= \int d^{10} Z \Big[\frac{1}{2} V_0 D^\alpha \overline{D}^2 D_\alpha V_0 - V_{-1} D^2 \overline{D}^2 V_1 - V_0 \, \partial D^2 V_{-1} - \\ &\qquad - V_1 \, \overline{\partial} \, \overline{D}^2 V_0 - \frac{1}{2} V_0 \, \partial \overline{\partial} \, V_0 \Big] \, . \end{aligned} \tag{5.17}$$

Under the identification $V \equiv V_0$, $\Omega \equiv i \overline{D}^2 V_1$, $\overline{\Omega} \equiv i D^2 V_{-1} = -i D^2 \overline{V}_1$, the action (5.17) coincides with (3.5) for the abelian case. Then, as expected, the action (5.15) describes the dynamics of an abelian $\mathcal{N} = (0,1)$ vector multiplet in six dimensions.

[3] To our knowledge the tropical multiplet action is not explicitly known in the nonabelian case. However, it is important to note in [6] a prescription was given to extract the entire nonabelian SYM action from the one written in harmonic superspace.

The minimal coupling of general SYM gauge multiplets to hypermultiplets in 6D projective superspace can be realized as in the 4D case [5, 4, 6]. For example, the action for a polar multiplet in the fundamental representation of the gauge group coupled to a tropical multiplet is given by

$$\int d^6x d^4\theta \left\{ \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta} \breve{\Upsilon} e^V \Upsilon \right\} . \tag{5.18}$$

The action is invariant under the gauge transformations

$$\Upsilon' = e^{i\Lambda}\Upsilon\,, \qquad \breve{\Upsilon}' = \breve{\Upsilon} e^{-i\breve{\Lambda}}\,, \qquad (e^V)' = e^{i\breve{\Lambda}} e^V e^{-i\Lambda}\,, \tag{5.19}$$

where the gauge parameters Λ and $\breve{\Lambda}$ are respectively artic and antartic projective superfields. The 6D projective superspace description of multiplets with opposite chirality goes along the same lines of the previous construction with ∂ exchanged with $-\overline{\partial}$.

We now address the issue of covariant quantization in 6D projective superspace. To this end we expect the equivalence between the $\mathcal{N} = (1,0)$ algebra written in the 4D formalism and the algebra of 4D $\mathcal{N} = 2$ supersymmetry with complex central charge should be still relevant in order to extend the 4D results to six dimensions.

As an example, we concentrate on the polar multiplet. In [5] the quantization of the 4D polar multiplet was performed in the case of an underlying $\mathcal{N} = 2$ supersymmetry with complex central charge m. Exploiting the formal identification[4] $m \leftrightarrow \partial$ we can easily argue the covariant propagator for the 6D (ant)artic superfield. It is in fact sufficient to take the result of [5] and rewrite it in six dimensions with the correct insertions of ∂ and $\overline{\partial}$. The propagator for the 6D $\mathcal{N} = (1,0)$ polar multiplet we expect then to be

$$\langle \breve{\Upsilon}(Z,\zeta)\Upsilon(Z',\zeta')\rangle = \frac{1}{\zeta'(\zeta-\zeta')^3} \frac{\nabla^2(\zeta)\overline{\nabla}^2(\zeta)\nabla^2(\zeta')\overline{\nabla}^2(\zeta')}{\Box_6}\, \delta^8(\theta-\theta')\delta^6(x-x')\,, \tag{5.20}$$

where $Z = (x_i, \theta^{a\alpha}, \overline{\theta}^{\dot{\alpha}}_a)$ are the coordinates of the 6D superspace and the $\nabla_\alpha(\zeta)$, $\overline{\nabla}_{\dot{\alpha}}(\zeta)$ have been defined in (5.5).

For the 4D tropical multiplet in $\mathcal{N} = 2$ projective superspace with central charge the quantization has not been performed yet. Therefore, we cannot exploit the 4D results to easily infer the form of its propagator in six dimensions. In any case, the covariant quantization in 6D projective superspace along the lines of [5] has yet to be rigorously developed.

So far we have restricted our attention to the polar and tropical multiplets as projective superspace extensions of the 6D CNM hypermultiplet and vector multiplet, respectively. In standard 4D projective superspace a larger class of multiplets has been studied and classified. The classification is based on the analiticity properties of the projective superfields in the ζ-plane. Projective superfields with a finite series expansion in ζ produce 4D complex $O(p)$ and real $O(2n)$ tensor multiplets. In the limits $p \to \infty$, $n \to \infty$ these give the polar and tropical multiplets, respectively [5, 4, 6]. Since our 6D projective superspace

[4]Note that this is the identification which was pointed out at the end of section 4 between the 6D ∂-derivative and the 4D complex mass m. The only difference is in the present context the mass plays the role of the central charge.

is essentially defined in the same way as the 4D one (in particular for what concerns the ζ-plane) it is clear the same class of tensor multiplets can be easily constructed also in the 6D case.

6. Conclusions and outlooks

In these notes, using a formalism which keeps manifest the 4D $\mathcal{N} = 1$ supersymmetry, we have introduced a new formulation of 6D $\mathcal{N} = 1$ hypermultiplet in terms of chiral-nonominimal (CNM) superfields. The CNM formulation is dual to the chiral-chiral (CC) description already present in the literature [3]. We have coupled the CC and CNM hypermultiplets to 6D SYM, covariantly with respect to the geometry of the 4D, $\mathcal{N} = 1$ superspace. We have studied in detail the superfield quantization of all the previous multiplets. Furthermore, we have developed a 6D projective superspace formalism in which the 6D CNM and vector multiplets naturally emerge. We have also discussed the covariant quantization of the (ant)artic projective superfields.

Armed with these results it would be interesting to investigate quantum properties of 6D supersymmetric models. The advantage of using a 4D, $\mathcal{N} = 1$ superfield formulation is in the possibility to compare diagrams which arise in the 6D case to the 4D analogues largely studied in the literature. This powerful technique has been already used in [12] to study one-loop properties of the ten dimensional $\mathcal{N} = 1$ SYM in correspondence to the four dimensional $\mathcal{N} = 4$ SYM. Through the introduction of a 6D projective superspace we have also established a formalism which could be even more efficient for exploring quantum properties of vector and hyper-multiplets in 6D. For example it might be possible to exploit these formalism to extend the study of 6D gauge anomalies á la previous work [22].

An interesting issue which might be worth studying in detail is the relation between 6D projective and harmonic superspaces [7], along the lines of [6] in the 4D case. From the harmonic superspace perspective, 6D is interesting being the highest dimension in which the powerful standard harmonic approach can be used. We expect the relation between 6D harmonic [23] and projective superspaces to have no relevant differences from the 4D case. The polar multiplet will be the projective superspace version of the q^+ hypermultiplet and the tropical multiplet will be related to the analytic harmonic gauge prepotential V^{++}. Our expectation is also supported by the results recently obtained in [16] for the 5D case. In six dimensions the only difference would be the fact, using the 4D spinor notation, the SUSY algebra turns out to have a complex central charge. However, this should not affect the structures which constrain the harmonics on one side and the ζ-complex-plane on the other one. Trying to understand the precise formulation of nonabelian SYM in 6D projective superspace from the harmonic one might be a useful indirect approach.

Recently in [24] using 6D harmonic superspace there was given the action of a renormalizable higher derivatives 6D SYM theory. It would be interesting to find the analogue of this theory written in 4D $\mathcal{N} = 1$ superfields formalism and projective superspace to study quantum properties of this model using our formalism.

Having a complete understading of the 6D harmonic superspace would be very useful for addressing many issues. An interesting question to investigate in this context would

JHEP05(2006)051

be how the "harmonic anomalies" which arise in 4D quantum theories manifest themselves in a 6D setting. Furthermore, six dimensional harmonic superspace might be the most efficient approach to analyze 6D $\mathcal{N}=1$ nonlinear sigma-models in a completely covariant way.

The topic of six dimensional nonlinear sigma-models is an intriguing one which has been not very well investigated to our knowledge. Since the construction of 6D hypermultiplets and SYM is efficiently developed using 4D $\mathcal{N}=1$ superfields as ingredients, it is natural to ask how to build supersymmetric sigma-models in this formalism and what are their geometric properties [25].

In this respect the 6D projective superspace formulation, rather than the harmonic one, should be the natural starting point. In fact, once the reduction of the projective superfields to their component superfields has been performed, we obtain an action which is written in terms of 4D CNM $\mathcal{N}=1$ superfields.[5] The six-dimensional Lorentz invariance of this action is not manifest. Therefore, one of the main questions we need answer is how Lorentz invariance gets restored once the model is reduced to the physical field components. This is a good starting point to attempt a formulation of supersymmetric CC sigma-models and CNM sigma-models which generalize the ones coming from projective superspace. 6D Lorentz invariance imposes non-trivial geometrical constraints on the sigma-model functions which describe the target space manifold and brings to hyper-Kähler geometries. To this regard the CNM-CC duality is a really interesting issue.

"*If you are out to describe the truth, leave elegance to the tailor.*"
— Albert Einstein

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A. 6D Weyl spinors

In this section we introduce our notations and conventions for 6D spinors.

In six dimensions, $(1,0)$ and $(0,1)$ Weyl spinors belong to the fundamental representation of SU*(4) and to the transpose representation, respectively. These representations can be decomposed into 4D spinor representations. Practically, a four component spinor index $\tilde{\alpha}$ of SU*(4) can be replaced by a pair of undotted and dotted indices $(\alpha,\dot{\alpha})$ of $\mathrm{Sl}(2,\mathbb{C})$ and a $(1,0)$ Weyl spinor can be written as

$$\Psi^{\tilde{\alpha}}=\begin{pmatrix}\psi_1^{\alpha}\\ \overline{\psi}_2^{\dot{\alpha}}\end{pmatrix}\;;\qquad \overline{\Psi}^{\tilde{\alpha}}=\begin{pmatrix}-\psi_2^{\alpha}\\ \overline{\psi}_1^{\dot{\alpha}}\end{pmatrix} \tag{A.1}$$

[5] A first example of how things should work can be found in [16] where the construction of 4D CNM sigma-models [21] has been generalized to five dimensions.

where $\overline{\Psi}^{\tilde{\alpha}} \equiv \mathcal{C}^{\tilde{\alpha}}{}_{\dot{\tilde{\beta}}} \left(\overline{\Psi}\right)^{\dot{\tilde{\beta}}}$ is the complex-conjugated of $\Psi^{\tilde{\alpha}}$ written in the left representation using the 6D charge-conjugation matrix $\mathcal{C}^{\tilde{\alpha}}{}_{\dot{\tilde{\beta}}} = \begin{pmatrix} 0 & -\delta^{\alpha}{}_{\dot{\beta}} \\ \delta^{\dot{\alpha}}{}_{\beta} & 0 \end{pmatrix}$.

The six-dimensional gamma matrices Γ^{μ} $\mu = 0, \ldots, 5$ acting on $(1,0)$ Weyl spinors can be represented as

$$\Gamma^{\mu}_{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} \Gamma^{\mu}_{\alpha\beta} & \Gamma^{\mu}_{\alpha\dot{\beta}} \\ -\Gamma^{\mu}_{\beta\dot{\alpha}} & \Gamma^{\mu}_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \tag{A.2}$$

with

$$\begin{array}{lll} \Gamma^{a}_{\alpha\dot{\beta}} = \sigma^{a}_{\alpha\dot{\beta}}\,, & \Gamma^{a}_{\alpha\beta} = \Gamma^{a}_{\dot{\alpha}\dot{\beta}} = 0\,, & (a = 0,1,2,3)\,; \\ \Gamma^{4}_{\alpha\dot{\beta}} = 0\,, & \Gamma^{4}_{\alpha\beta} = iC_{\alpha\beta}\,, & \Gamma^{4}_{\dot{\alpha}\dot{\beta}} = iC_{\dot{\alpha}\dot{\beta}}\,; \\ \Gamma^{5}_{\alpha\dot{\beta}} = 0\,, & \Gamma^{5}_{\alpha\beta} = C_{\alpha\beta}\,, & \Gamma^{5}_{\dot{\alpha}\dot{\beta}} = -C_{\dot{\alpha}\dot{\beta}}\,, \end{array} \tag{A.3}$$

$\sigma^{a}_{\alpha\dot{\beta}}$ being the Pauli matrices and $C_{\alpha\beta} = C_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Using the SU*(4) invariant $\epsilon^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}$ ($\epsilon^{1234} = \epsilon_{1234} = 1$) it is possible to raise and lower pairs of antisymmetric indices. In particular, the gamma matrices $\Gamma^{\mu\tilde{\alpha}\tilde{\beta}}$ acting on a $(0,1)$ Weyl spinor $\Psi_{\tilde{\alpha}}$ are given by

$$\Gamma^{\mu\tilde{\alpha}\tilde{\beta}} = \frac{1}{2}\epsilon^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\Gamma^{\mu}_{\tilde{\gamma}\tilde{\delta}}\,, \qquad \Gamma^{\mu}_{\tilde{\alpha}\tilde{\beta}} = \frac{1}{2}\epsilon_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\Gamma^{\mu\tilde{\gamma}\tilde{\delta}}\,. \tag{A.4}$$

These matrices satisfy

$$\Gamma^{\mu}_{\tilde{\alpha}\tilde{\beta}}\Gamma^{\nu\tilde{\beta}\tilde{\gamma}} + \Gamma^{\nu}_{\tilde{\alpha}\tilde{\beta}}\Gamma^{\mu\tilde{\beta}\tilde{\gamma}} = -2\eta^{\mu\nu}\delta^{\tilde{\gamma}}_{\tilde{\alpha}}\,, \qquad \Gamma^{\mu}_{\tilde{\alpha}\tilde{\beta}}\Gamma^{\tilde{\gamma}\tilde{\delta}}_{\mu} = 4\delta^{[\tilde{\gamma}}_{\tilde{\alpha}}\delta^{\tilde{\delta}]}_{\tilde{\beta}}\,. \tag{A.5}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,\cdots,1)$.

Introducing the spacetime derivatives $\partial_{\alpha\dot{\beta}} \equiv \sigma^{a}_{\alpha\dot{\beta}}\partial_a$ and $\partial \equiv (\partial_4 - i\partial_5)$, $\overline{\partial} \equiv (\partial_4 + i\partial_5)$ we have

$$\partial_{\tilde{\alpha}\tilde{\beta}} \equiv \Gamma^{\mu}_{\tilde{\alpha}\tilde{\beta}}\partial_{\mu} = \begin{pmatrix} iC_{\alpha\beta}\partial & \partial_{\alpha\dot{\beta}} \\ -\partial_{\beta\dot{\alpha}} & iC_{\dot{\alpha}\dot{\beta}}\overline{\partial} \end{pmatrix}, \qquad \partial^{\tilde{\alpha}\tilde{\beta}} \equiv \Gamma^{\mu\tilde{\alpha}\tilde{\beta}}\partial_{\mu} = \begin{pmatrix} -iC^{\alpha\beta}\overline{\partial} & \partial^{\alpha\dot{\beta}} \\ -\partial^{\beta\dot{\alpha}} & -iC^{\dot{\alpha}\dot{\beta}}\partial \end{pmatrix}. \tag{A.6}$$

The action which describes the free dynamics of a six-dimensional $(1,0)$ Weyl spinor is

$$\int d^6x \left[\overline{\Psi}^{\tilde{\alpha}}\Gamma^{\mu}_{\tilde{\alpha}\tilde{\beta}} i\partial_{\mu}\Psi^{\tilde{\beta}}\right] = \int d^6x \left[-\overline{\psi}_1^{\dot{\alpha}} i\partial_{\alpha\dot{\alpha}}\psi_1^{\alpha} - \overline{\psi}_2^{\dot{\alpha}} i\partial_{\alpha\dot{\alpha}}\psi_2^{\alpha} - \psi_2^{\alpha}\partial\psi_{1\alpha} - \overline{\psi}_2^{\dot{\alpha}}\overline{\partial}\overline{\psi}_{1\dot{\alpha}}\right], \tag{A.7}$$

whereas for a $(0,1)$ Weyl spinor we have

$$\int d^6x \left[\overline{\Psi}_{\tilde{\alpha}}\Gamma^{\mu\tilde{\alpha}\tilde{\beta}} i\partial_{\mu}\Psi_{\tilde{\beta}}\right] = \int d^6x \left[-\overline{\psi}_1^{\dot{\alpha}} i\partial_{\alpha\dot{\alpha}}\psi_1^{\alpha} - \overline{\psi}_2^{\dot{\alpha}} i\partial_{\alpha\dot{\alpha}}\psi_2^{\alpha} + \psi_2^{\alpha}\overline{\partial}\,\psi_{1\alpha} + \overline{\psi}_2^{\dot{\alpha}}\partial\,\overline{\psi}_{1\dot{\alpha}}\right]. \tag{A.8}$$

Given the structure (A.6) for the 6D spacetime derivatives the action (A.8) is simply obtained from (A.7) by the exchange $\partial \leftrightarrow -\overline{\partial}$.

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